Foundations of Statistics I: Decision Theory
Problem Set 2.

Problem 1 (20 pts) *From the 2005 final exam, one of two. Consider a point estimation problem in which you observe $x_1, \ldots, x_n$ as iid random variables from the Poisson distribution*

$$p(x|\theta) = \frac{1}{x!}\theta^x e^{-\theta}.$$

Assume a squared error of estimation loss $L(\theta, a) = (a - \theta)^2$, and assume a prior distribution on θ given by the Gamma density

$$\pi(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\theta/\beta}.$$

1. *Show that the Bayes decision rule with respect to the prior above is of the form*
$$\delta^*(x_1, \ldots, x_n) = a + b\bar{x}$$
where $a > 0$, $b \in (0, 1)$ and $\bar{x} = 1/n \sum_i x_i$. You may use the fact that the distribution of $\sum_i x_i$ is a Poisson with parameter $n\theta$ without proof.
2. *Compute and graph the risk functions of $\delta^*(x_1, \ldots, x_n)$ and that of the MLE $\delta(x_1, \ldots, x_n) = \bar{x}$*
3. *Compute the Bayes risk of $\delta^*(x_1, \ldots, x_n)$ and show that it is a) decreasing in n and b) it goes to 0 as n gets large.*
4. *Suppose an investigator wants to collect a sample that is large enough that the Bayes risk after the experiment is half of the Bayes risk before the experiment. Write the nonlinear equation*
$$A(n, \alpha, \beta) = 0$$
that you would need to solve for n to find that sample size.

Problem 2 *[10 points] Suppose that X is a $Bin(n, \theta)$ and that*

$$L_0(\theta, a) = \frac{(\theta - a)^2}{\theta(1 - \theta)}.$$

Show that the Bayes decision rule with respect to a uniform prior has constant risk function R. Now, consider the quadratic loss function $L_1(\theta, a) = (\theta - a)^2$ and assume that θ has a priori $Beta(\alpha, \beta)$ distribution. Find α and β such that the Bayes decision rule has constant risk R. Revisit this example after we will have discussed the relationship between Bayes and minimax rules.

Problem 3 (10 pts) *Prove that the Bayes rule under absolute error loss is the posterior median. More precisely, suppose that $E(|\theta|) < \infty$. Show that a number a^* satisfies*

$$E(|\theta - a^*|) = \inf_{-\infty < a < \infty} E(|\theta - a|)$$

if and only if a^* is a median of the distribution of θ.